Def: R is semiprimitive (= Jacobson semisimple, J-semisimple) if $J(R) = \underline{0}$.

Lemma 3.3: If $I \trianglelefteq R$, $I \subseteq J(R)$, then $J(R/I) = J(R)/I$.

In particular: $J(R/J(R)) = \underline{0}$.

Proof: Clear using $J(R) = \bigcap \{\tilde{I} : \tilde{I} \leq R_R \text{ maximal}\}$. $\square$

Prop 3.4: 1) R and $R/J(R)$ have the same simple modules

2) $r \in R$ is [right] invertible $\Leftrightarrow$ $r + J(R) \in \overline{R} := R/J(R)$ is [right] invertible

Recall: If $I \trianglelefteq R$, there is a bijection

$$\{M \in \text{Mod-}R : I \subseteq \text{ann}(M)\} \longleftrightarrow \text{Mod-}R/I$$

"$\mapsto$": $m(r+I) := mr$ (works because $mI = 0$)

"$\leftarrow$": $mr := m(r+I)$

Respects submodules, morphisms, etc. (category isomorphism)

Proof: 1) By the remark on Mod-R/I & definition of $J(R)$

2) "$\Rightarrow$" ✓ "$\Leftarrow$" $r + J(R)$ right invertible $\Rightarrow$ $\exists s \in R$

$\Rightarrow 1 + J(R) = (r + J(R))(s + J(R)) = rs + J(R)$

$\Rightarrow \exists x \in J(R)$: $1 = rs - x \Rightarrow rs = 1 + x \overset{L3.1}{\Rightarrow} \exists y : r(sy) = 1$ $\square$

3.1 Nil and Nilpotent Ideals

Def: A right ideal $I \leq R$ is

- nilpotent if $\exists n \geq 0$: $I^n = \underline{0}$ $(\forall a_1, \dots, a_n \in I: a_1 \cdots a_n = 0)$
- nil if every element of I is nilpotent $(\forall a \in I \, \exists n \geq 0: a^n = 0)$

Nilpotent $\Rightarrow$ nil, but

Exm: In $\mathbb{Z}[x_1, x_2, x_3, \dots] / (x_1, x_2^2, x_3^3, \dots)$, $I = (\bar{x}_1, \bar{x}_2, \dots)$

is nil, not nilpotent

⚠ a, b nilpotent $\not\Rightarrow$ $a+b$ nilpotent in nc rings! (E.g. in $M_2(K)$: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$)

Remark: Easy to prove: $I, J \leq R_R$ nilpotent $\Rightarrow$ $I+J$ nilpotent.

Open Conjecture (Köthe Conjecture): $I, J \leq R_R$ nil $\Rightarrow$ $I+J$ nil.

Lemma 3.5: If $I \leq R_R$ is nil, then $I \subseteq J(R)$

Proof: Let $x \in I \Rightarrow \forall r \in R$: xr nilpotent, say $(xr)^n = 0$

$$\Rightarrow (1 - xr) \cdot (1 + xr + (xr)^2 + \dots + (xr)^{n-1}) = 1 - (xr)^n = 1$$

$\Rightarrow 1 - xr$ (right) invertible $\Rightarrow x \in J(R)$. $\square$

⚠ Not every nilpotent element is contained in $J(R)$.

E.g. $R = M_n(D)$, D div. ring $\Rightarrow J(R) = \underline{0}$, but there are nilpotent matrices!

Thm 3.6: If R is right artinian, $J(R)$ is nilpotent.

In particular: 1) $J(R)$ is the largest nilpotent right [left] ideal

2) If $I \leq R_R$ is nil, then I is nilpotent

Proof: Suffices to show $J(R)$ nilpotent, then L3.5 gives 1) & 2).

The chain $J(R) \supseteq J(R)^2 \supseteq J(R)^3 \supseteq \dots$ stabilizes

$\rightarrow \exists I \trianglelefteq R$: $I = J(R)^k$ for sufficiently large k.

Note: $I^2 = I$

Claim: $I = 0$.

Suppose $I \neq 0$. Let $A \leq R_R$ be a right ideal minimal with the property $AI \neq 0$ (exists, because R is right ordinian).

Let $x \in A$ s.t. $xI \neq 0$

$$\Rightarrow xI \cdot I = xI^2 = xI \neq 0 \underset{xI \subseteq A}{\Longrightarrow} A = xI$$

$$\Rightarrow x = xy \text{ with } y \in \underset{\cong J(R)}{I} \Longrightarrow x\underbrace{(1-y)}_{\in R^\times} = 0 \Rightarrow x = 0 \quad ⚡.$$

$\square$

3.2 Semisimple rings again

Lemma 3.7 A right ideal $I \leq R_R$ is a direct summand of R_R
$\Leftrightarrow$ $I = eR$ with e idempotent

Proof: „$\Leftarrow$": $\forall r \in R$: $r = er + (1-e)r$, so $R_R = eR + (1-e)R$.

If $r \in eR \cap (1-e)R \Rightarrow r = ex = (1-e)y \quad (x, y \in R)$

$$\Rightarrow er = e^2x = ex = r, \text{ so } r = er = \underbrace{e(1-e)}_{= e - e^2 = 0}y = 0$$

„$\Rightarrow$": Suppose $R_R = I_R \oplus J_R \Rightarrow 1 = e + (1-e)$ with $e \in I$, $(1-e) \in J$

$$\Rightarrow e(1-e) = (1-e)e \in I \cap J \Rightarrow 0 = e(1-e) = (1-e)e$$

$$\Rightarrow e = e^2.$$

Finally $\forall x \in I$: $\overset{I}{\underset{\cup}{x}} = \overset{I}{\underset{\cup}{e}}x + \underbrace{(1-e)x}_{\in J \cap I = 0} \overset{(1-e)x = 0}{\Longrightarrow} x = ex$, so $I = eR$.

$\square$

Thm 3.8 TFAE for a ring R

(a) R is semisimple

(b) R is semiprimitive ($J(R)=0$) and right artinian

(c) R — " ——— and satisfies the DCC on principal right ideals

Proof: (a) $\Rightarrow$ (b) $\quad R_R$ has finite length, so is right artinian.

$\exists I \leq R_R : \quad R_R = J(R) \oplus I \quad$ [T2.12(c)]

If $J(R) \neq 0$, there exists a max. right ideal $M \supseteq I$ (Zorn's lemma)

$\overset{M \subsetneq R}{\Longrightarrow} J(R) \nsubseteq M$ ↯.

(b) $\Rightarrow$ (c) ✓

(c) $\Rightarrow$ (a)

Note: (i) Every $0 \neq I \leq R_R$ contains a minimal nonzero right ideal

[Take a minimal element of $\{xR : x \in I, xR \neq 0\}$ by DCC.]

(ii) Every minimal $0 \neq I \leq R_R$ is a direct summand of R_R

[$I \neq \underline{0}$, but $J(R)=0 \Rightarrow \exists$ maximal right ideal $M \leq R_R$ s.t.

$I \nsubseteq M \Rightarrow I + M = R$. Since I is simple, $I \cap M = \underline{0}$]

We now find $0 \neq A_1, \dots, A_n$ minimal right ideals s.t. $R_R = A_1 \oplus \dots \oplus A_n$.

Suppose $A_1, \dots, A_{n-1}$ ($n \geq 1$) have been constructed s.t. $R_R = A_1 \oplus \dots \oplus A_{n-1} \oplus B_n$,

$B_n \leq R_R$. Can assume $B_n \neq 0$. Let $0 \neq A_n \leq B_n$ be minimal (using (i)).

Then $R_R = A_n \oplus C_n$ (by (ii)), and $B_n = A_n \oplus \underbrace{(B_n \cap C_n)}_{=: B_{n+1} \subsetneq B_n}$

This gives A_n. Note $B_1 \supsetneq B_2 \supsetneq \ldots$ is a chain of principal right ideals [L3.7], so by DCC it stabilizes. But this only happens when $B_n = \underline{0}$. $\square$

<u>Cor 3.9:</u> If R is right artinian (e.g. a f.-d. algebra over a field), then R semisimple $\Leftrightarrow$ $J(R) = \underline{0}$.

<u>Cor 3.10</u> R right artinian $\Rightarrow$ $R/J(R)$ semisimple and $J(R)$ nilpotent

<u>Strategy:</u> Consider $J(R)^i / J(R)^{i+1}$ for $i = 0, 1, \ldots, n$

<u>Def:</u> R is semiprimary if $R/J(R)$ is semisimple and $J(R)$ is nilpotent

So: right [left] artinian $\Rightarrow$ semiprimary

<u>Thm 3.11</u> (Hopkins-Levitzki) If R is semiprimary and $M \in \text{Mod-}R$, TFAE:
(a) M is noetherian
(b) M is artinian
(c) M has a composition series (i.e. $\ell(M) < \infty$)

<u>Proof:</u> (c) $\Rightarrow$ (a), (b) by [L2.9].

<u>(a), (b) $\Rightarrow$ (c):</u> Let $n \geq 0$ s.t. $J(R)^n = \underline{0}$, $\overline{R} := R/J(R)$.

Consider $M \supseteq MJ(R) \supseteq MJ(R)^2 \supseteq \ldots \supseteq MJ(R)^n = \underline{0}$.

Suffices to show: $MJ(R)^i / MJ(R)^{i+1}$ has a composition series ($i = 0, \ldots, n-1$)

Now $MJ(R)^i / MJ(R)^{i+1}$ is a module over the semisimple $\overline{R}$,

hence a direct sum of simple $\overline{R}$-modules. Since M is noetherian

or ardinian, so is $MJ(R)^i / MJ(R)^{i+1}$, so the direct sum is finite.

Thus $MJ(R)^i / MJ(R)^{i+1}$ has a composition series as $\overline{R}$-module,

and this is also a composition series as R-module. $\square$

Cor 3.12: R right ardinian $\iff$ R right noetherian and semiprimary

Proof: „$\Leftarrow$" Apply T3.11 to R_R

„$\Rightarrow$" R right ardinian $\Rightarrow$ R semiprimary; Again apply T3.11 to R_R. $\square$

In particular: R right ardinian $\Longrightarrow$ R right noetherian.

Cor 3.13 If R_R is right ardinian and $M \in \text{Mod-}R$ is finitely generated, then $\ell(M) < \infty$.

3.3 Nakayama's lemma

Lemma 3.14 (Nakayama) If $M \in \text{Mod-}R$ is f.g. and $MJ(R) = M$, then $M = 0$.

Proof: Let $M = \langle m_1, \dots, m_k \rangle_R$ with k minimal. If $k \geq 1$,

can write $m_k = m_1 r_1 + \dots + m_{k-1} r_{k-1} + m_k r_k$ with $r_i \in J(R)$.

$\Rightarrow$ $m_k(1 - r_k) \in \langle m_1, \dots, m_{k-1} \rangle_R$. Since $1 - r_k \in R^\times$, also

$m_k \in \langle m_1, \dots, m_{k-1} \rangle_R$ $\lightning$ k minimal. $\square$

Cor 3.15 If $M \in \text{Mod-}R$, $N \leq M$ s.t. M/N is f.g., and $M = N + MJ(R)$, then $M = N$.

Proof: Apply L3.14 to M/N.

An application: Let $M \in \text{Mod-}R$. Then $\overline{M} := M/MJ(R)$ is annihilated by $J(R)$, and hence naturally a $\overline{R} := R/J(R)$-module.

Cor 3.16: If M is f.g., and $\{x_i\}_{i\in I}$ is a family in M s.t. $\{\overline{x_i}\}_{i\in I}$ generates $\overline{M}$ (over $\overline{R}$), then $\{x_i\}_{i\in I}$ generates M.

Proof: Apply C3.15 to $N = \langle x_i : i \in I\rangle_R$. $\square$

If $f: M \to M'$ is an R-hom, then $f(MJ(R)) \subseteq M'J(R)$, so it induces $\overline{f}$:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & M' \arrow[d] & (\text{Mod-}R) \\ \overline{M} \arrow[r, "\overline{f}"] & \overline{M'} & (\text{Mod-}\overline{R}) \end{tikzcd}$$

Cor 3.17: If $M' \in \text{Mod-}R$ is f.g. and $f: M \to M'$ is s.t. $\overline{f}: \overline{M} \to \overline{M'}$ is surjective, then $f: M \to M'$ is surjective.

Proof: $\overline{f}(\overline{M}) = \overline{M}' \to M' = f(M) + M'J(R) \Rightarrow M' = f(M)$. $\square$

3.4 Group Algebras (w/o proof)

K field. If G is infinite, then $K[G]$ is not semisimple [P2.24].

Thm 3.18 (Amitsur) Let K be a non-algebraic field ext. of $\mathbb{Q}$. Then $K[G]$ is semiprimitive for any group.

($K = \mathbb{C}$ .. Rickard 1950; full thm. by Amitsur 1959).

For a proof: [Lam01, (6.12)]

For fields of algebraic numbers (e.g. $K = \mathbb{Q}$), this is still open.

Difficult problems on group algebras (Kaplansky's Conjectures, 50s):

If G is torsion-free, then

- $K[G]$ is a domain (zero divisor conjecture) $\to$ OPEN
- $K[G]$ does not contain non-trivial idempotents (idempotent conj.) $\to$ OPEN
- $K[G]$ contains only trivial units ($K[G]^\times = K^\times G$) (unit conj.) $\to$ FALSE

Gardam 2021: Counterexample to unit conjecture w. $K = \mathbb{F}_2$

2023: Same counterexample w. char $K = 0$.

4. Structure of Primitive Rings & Jacobson Density Theorem

$$\begin{array}{ccc}
M_{n_1}(D_1)\times\dots\times M_{n_k}(D_k) & & J(R)=0 \\
\text{semisimple} & \Longrightarrow & \text{semiprimitive} \\
\Uparrow & & \Uparrow \\
\text{simple artinian} & \Longrightarrow & \text{right primitive} \\
M_n(D) & & \text{faithful simple right [left] module}
\end{array}$$

Note: Semisimple, semiprimitive, simple artinian are left-right symmetric. Left primitive $\not\Leftrightarrow$ right primitive [Bergman 1964]

For any ring R, $R/J(R)$ is semiprimitive

Prop 4.1 Every semiprimitive ring R is a subdirect product of right primitive rings, i.e., $R \hookrightarrow \prod_{i \in I} R_i$ (injective) with R_i right primitive, and all projections $R \to R_i$ surjective.

Proof: $0 = J(R) = \bigcap_{\substack{M \in \mathrm{Mod}\text{-}R \\ M \text{ simple}}} \mathrm{ann}(M_R)$

$$\Rightarrow R \hookrightarrow \prod_{M_R \text{ simple}} \underbrace{R/\mathrm{ann}(M)}_{\text{right primitive}}, \quad \text{and } R \to R/\mathrm{ann}(M) \text{ is surjective}$$

$\square$

Sometimes a useful strategy: prove something for right primitive rings, lift it to subdirect products ($\to$ semiprimitive rings), then modulo the Jacobson radical ($\to$ arbitrary rings; usually hard or impossible).